OTTICA Q 18/11/20

$$\chi(\lambda, s) \stackrel{def}{=} \langle D(\lambda) \rangle e^{\frac{s}{2}|\lambda|^2}$$

$\chi(\lambda, s=+1)$ è fn caratter di $P(\alpha)$:

$$\chi(\lambda, s=+1) = Tr\left[\rho\, e^{\lambda a^+} e^{-\lambda^* a}\right] =$$

$$Tr\Big[\underbrace{\int d^2\alpha\, P(\alpha) |\alpha\rangle\langle\alpha|}_{\rho} e^{\lambda a^+}\Big] = \int d^2\alpha\, P(\alpha)\, Tr\left[e^{-\lambda^* a} |\alpha\rangle\langle\alpha| e^{\lambda a^+}\right]$$

$$= \int d^2\alpha\, P(\alpha)\, e^{-\lambda^*\alpha + \lambda\alpha^*}\, \overbrace{Tr\left[|\alpha\rangle\langle\alpha|\right]}^{=1} \leftarrow TF \text{ di } P(\alpha)$$

$\chi(\lambda, s=-1)$ è fn caratt di $Q(\alpha)$

$$\chi(\lambda, s=-1) = Tr\left[\rho\, e^{-\lambda^* a} e^{\lambda a^+}\right] =$$

$$= \int \frac{d^2\alpha}{\pi} \langle\alpha| e^{\lambda a^+} \rho\, e^{-\lambda^* a} |\alpha\rangle = \int d\alpha\, e^{\lambda\alpha^* - \lambda^*\alpha} \underbrace{\frac{\langle\alpha|\rho|\alpha\rangle}{\pi}}_{=Q(\alpha)}$$

Fn di Wigner generalizzata

def: antiTF della Fn caratteristica $\chi(\lambda, s)$

$$\boxed{W(\alpha, s) \stackrel{def}{=} \int \frac{d^2\lambda}{\pi^2} \underbrace{e^{\alpha\lambda^* - \alpha^*\lambda} \, e^{\frac{s}{2}|\lambda|^2} \langle D(\lambda) \rangle}_{\chi(\lambda, s)}}$$

$\Rightarrow W(\alpha, s=1) = P(\alpha)$, $W(\alpha, s=-1) = Q(\alpha)$

Fn di Wigner $\stackrel{def}{=} W(\alpha, s=0)$

ordinam simmetrico $:f(a,a^\dagger):_S \stackrel{def}{=}$

$$\frac{1}{2}\left(:f(a,a^\dagger): + :f(a,a^\dagger):_A\right)$$

Fn di Wigner nello spazio delle fasi, p, q

$$\underbrace{W(q,p) = \frac{2}{\pi\hbar}\int dy\; e^{-2iyp/\hbar}\langle q-y|\rho|q+y\rangle}$$

autost della posiz $\hat{q}$

è uguale a quella introdotta sopra

anti TF della fn caratteristica $\chi(\lambda, s=0)$

$$\hat{a} = \frac{\hat{q}+i\hat{p}}{\sqrt{2\hbar}} \quad (\text{senza } \omega!) \qquad q = (a+a^\dagger)\sqrt{\frac{\hbar}{2}}$$

$$p = -i(a-a^\dagger)\sqrt{\frac{\hbar}{2}}$$

PROPRIETA' di $W(\alpha)$

NORMALIZZATA:

$$\int d^2\alpha \; W(\alpha, s) = 1$$

$$\int d^2\alpha \int \frac{d^2\lambda}{\pi^2}\, \underbrace{e^{\alpha\lambda^* - \alpha^*\lambda}}_{=\delta(\lambda)} \langle D(\lambda)\rangle e^{-\frac{s}{2}|\lambda|^2} = \int d^2\lambda\; \delta(\lambda)\,\times$$

$$\times \langle D(\lambda)\rangle \cancel{e^{-\frac{s}{2}|\lambda|^2}} = 1$$

$$= \mathrm{Tr}[\rho\; \cancel{e^{\lambda a^\dagger - \lambda^* a}}] = 1$$

NON è in generale positiva $W(\alpha, s) \not\geq 0$

PSEUDO PROBABILITA'

Wigner è una rappresentaz dello stato: c'è una relaz biunivoca tra $W(\alpha, s)$ e ρ

" " " $W(\alpha, s, O)$ e O

al posto di ρ

$$W(\alpha,s,O) \stackrel{def}{=} \int \frac{d^2\lambda}{\pi^2} e^{\alpha\lambda^* - \lambda\alpha^* + \frac{s}{2}|\lambda|^2} \mathrm{Tr}\left[O D(\lambda)\right]$$

Wigner da $\hat{O}$

viceversa O dalla Wigner:

$$\hat{O} \stackrel{!}{=} \int \frac{d^2\lambda}{\pi} d^2\alpha \, W(\alpha, s, \hat{O}) \, e^{\lambda\alpha^* - \lambda^*\alpha - \frac{s}{2}|\lambda|^2} D^\dagger(\lambda)$$

partiamo dall'identità

$$\boxed{\hat{O} = \int \frac{d^2\lambda}{\pi} \mathrm{Tr}\left[O D(\lambda)\right] D^\dagger(\lambda)}$$

identità tomografica

Wigner è l'anti TF di $\mathrm{Tr}\left[O D(\lambda)\right] e^{\frac{s}{2}|\lambda|^2}$

$$\Rightarrow \mathrm{Tr}\left[O D(\lambda)\right] = e^{-\frac{s}{2}|\lambda|^2} \int d^2\alpha \, e^{\lambda\alpha^* - \lambda^*\alpha} W(\alpha, s, \hat{O})$$

$$\hat{O} = \int \frac{d^2\lambda}{\pi} \mathrm{Tr}\left[O D(\lambda)\right] D^\dagger(\lambda) =$$

$$= \int \frac{d^2\lambda}{\pi} d^2\alpha \, e^{\lambda\alpha^* - \lambda^*\alpha - \frac{s}{2}|\lambda|^2} W(\alpha, s, O) D^\dagger(\lambda)$$

IDENTITÀ TOMOGRAFICA $\hat{O} = \int \frac{d^2\lambda}{\pi} \mathrm{Tr}\left[O D(\lambda)\right] D^\dagger(\lambda)$

nello spazio di Hilbert degli operatori $\Rightarrow$ il prod scalare tra vettori è il prod scalare di Hilbert-Schmidt

$$\langle \hat{A} | \hat{B} \rangle = \mathrm{Tr}[A^\dagger B] \qquad D(\lambda) = D^\dagger(-\lambda) = e^{\lambda a^\dagger - \lambda^* a}$$

$$\hat{O} = \int \frac{d^2\lambda}{\pi} \mathrm{Tr}[O D^\dagger(-\lambda)] D(-\lambda) \underset{\gamma = -\lambda \quad d^2\gamma = d^2\lambda}{=}$$

$$= \int \frac{d^2\gamma}{\pi} \mathrm{Tr}[D^\dagger(\gamma) O] D(\gamma) =$$

$$= \int d^2\gamma \, \Big| \frac{\hat{D}(\gamma)}{\sqrt{\pi}} \Big\rangle \Big\langle \frac{\hat{D}(\gamma)}{\sqrt{\pi}} \Big| \hat{O} \Big\rangle = |\hat{O}\rangle$$

notaz di Dirac per sp Hilbert operatori

CNES per base $\quad |x\rangle$ base $\Leftrightarrow \int dx \, |x\rangle\langle x| = \mathbb{1}$

$\Rightarrow \quad \dfrac{D(\gamma)}{\sqrt{\pi}}$ è una base ortonormale per lo spazio degli operatori

dimostro identità tomog usando l'overcompletezza dei coerenti

$$\underset{\hat{O}}{|\alpha\rangle\langle\alpha|} = \int \frac{d^2\lambda}{\pi} \mathrm{Tr}[|\alpha\rangle\langle\alpha| D(\lambda)] D^\dagger(\lambda)$$

$$\int \frac{d^2\lambda}{\pi} \mathrm{Tr}[|\alpha\rangle\langle\alpha| D(\lambda)] D^\dagger(\lambda) = \int \frac{d^2\lambda}{\pi} \langle\alpha| \underbrace{D(\lambda)|\alpha\rangle}_{= |\alpha+\lambda\rangle e^{\frac{1}{2}(\lambda\alpha^* - \lambda^*\alpha)}} D^\dagger(\lambda)$$

$$\langle\alpha|\alpha+\lambda\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\alpha+\lambda|^2}{2}} e^{\alpha^*(\alpha+\lambda)}$$

$$|\alpha+\lambda|^2 = |\alpha|^2 + |\lambda|^2 + \alpha\lambda^* + \alpha^*\lambda$$

$$= \int \frac{d^2\lambda}{\pi} e^{-\frac{|\lambda|^2}{2} - \alpha\lambda^* + \alpha^*\lambda} \underbrace{D^\dagger(\lambda)}_{= e^{-\lambda a^\dagger + \lambda^* a}} = \int \frac{d^2\lambda}{\pi} e^{-\frac{|\lambda|^2}{2}} \times$$

$$e^{-\lambda\underbrace{(a^\dagger - \alpha^*)} + \lambda^*(a-\alpha)}$$

$D(\alpha) a D^\dagger(\alpha)$

$$= \int \frac{d^2\lambda}{\pi} e^{-\frac{|\lambda|^2}{2}} D(\alpha) D^\dagger(\lambda) D(\alpha) \qquad = |0\rangle\langle 0|$$

$$D(\alpha) a D^\dagger(\alpha) = a - \alpha \qquad e^{B} A e^{-B} = A + [B,A] + \dots$$

hc di ambo i m

$$D(\alpha) a^\dagger D^\dagger(\alpha) = a^\dagger - \alpha^*$$

$$= D(\alpha) |0\rangle\langle 0| D^\dagger(\alpha) = |\alpha\rangle\langle\alpha|$$

$$\int \frac{d^2\lambda}{\pi} e^{-\frac{|\lambda|^2}{2}} D^\dagger(\lambda) \overset{BCH}{=} \int \frac{d^2\lambda}{\pi} e^{-|\lambda|^2} e^{-\lambda a^\dagger} e^{\lambda^* a} =$$

$$= \int \frac{d^2\lambda}{\pi} e^{-|\lambda|^2} \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(-\lambda)^n}{n!} \frac{\lambda^{*m}}{m!} a^{\dagger n} a^m \overset{\lambda = \rho e^{i\varphi}}{=}$$

$\lambda^n \lambda^{*m}$

$$= \int_0^{2\pi} \frac{d\varphi}{\pi} \int_0^{+\infty} d\rho\, \rho\, e^{-\rho^2} \sum_{n,m} \frac{(-1)^n}{n!\,m!} \rho^{n+m} e^{i\varphi(n-m)} a^{\dagger n} a^m$$

$$= \delta_{nm} \frac{2\pi}{\pi}$$

$$= \int_0^{+\infty} \underbrace{d\rho\, 2\rho}_{d\rho^2}\, e^{-\rho^2} \sum_n \frac{(-1)^n}{(n!)^2} \rho^{2n} a^{\dagger n} a^n = \sum_{n=0}^{\infty} \frac{(-1)^n a^{\dagger n} a^n}{n!}$$

Γ di Eulero

$= n!$

$$= :e^{-a^\dagger a}: \overset{?}{=} |0\rangle\langle 0|$$

$$:e^{-a^\dagger a}: = \sum_{j k=0}^{\infty} |j\rangle\langle j| \sum_n \frac{(-1)^n}{n!} a^{\dagger n} \underbrace{a^n |k\rangle\langle k|}_{=\mathbb{1}} =$$

$$\langle j|a^{\dagger n} = \langle j-n| \frac{\sqrt{j!}}{\sqrt{(j-n)!}} \quad \text{se } n \le j; \qquad 0 \quad n > j$$

$$a^n|k\rangle = \frac{\sqrt{k!}}{\sqrt{(k-n)!}} |k-n\rangle \quad \text{se } n \le k; \qquad 0 \quad n > k$$

$$= \sum_{jk=0}^{\infty} \sum_{n=0}^{\min(j,k)} |j\rangle \underbrace{\langle j-n| k-n\rangle}_{\delta_{jk}} \langle k| \frac{(-1)^n}{n!} \frac{\sqrt{k!\,j!}}{\sqrt{(k-n)!\,(j-n)!}}$$

$$= \sum_{j=0}^{\infty} \sum_{n=0}^{j} |j\rangle\langle j| \frac{(-1)^n}{n!} \frac{j!}{(j-n)!}$$

$$= \sum_{j=0}^{\infty} |j\rangle\langle j| \sum_{n=0}^{j} \binom{j}{n} (-1)^n 1^{j-n} = (1-1)^j = 0^j = \delta_{j0}$$

$$(a+b)^j = \sum_{n=0}^{j} \binom{j}{n} a^j b^{n-j}$$

$$= |0\rangle\langle 0|$$

Proprietà Wigner

Valori di aspettazione direttamente dalla Wigner

$$\langle O \rangle = \mathrm{Tr}[\rho O] = \pi \int d^2\alpha \, W(\alpha, s) \, W(\alpha, -s, \hat{O})$$

$$= \pi \int d^2\alpha \, W(\alpha) \, W(\alpha, \hat{O})$$

$$\mathrm{Tr}[\rho O] = \mathrm{Tr}\Big[\underbrace{\int \frac{d^2\alpha \, d^2\lambda \, d^2\alpha' \, d^2\lambda'}{\pi^2} W(\alpha) \, D^\dagger(\lambda)}_{\rho} W(\alpha', \hat{O}) \, D^\dagger(\lambda')$$

$$e^{\lambda\alpha^* - \alpha\lambda^*} \; e^{\lambda'\alpha'^* - \alpha'\lambda'^*} \Big] =$$

$$= \int \frac{d^2\alpha \, d^2\lambda \, d^2\alpha' \, \cancel{d^2\lambda'}}{\pi^{\cancel{4}}} \, e^{\lambda\alpha^* - \alpha\lambda^* + \lambda'\alpha'^* - \alpha'\lambda'^*} \, \underbrace{Tr\left[D^+(\lambda) D^+(\lambda')\right]}_{\delta(\lambda+\lambda')}$$

$$\lambda' = -\lambda \qquad \lambda'^* = -\lambda^*$$

$$W(\alpha) \; W(\alpha' \, 0)$$

$$Tr\left[D^+(\lambda) D^+(\lambda')\right] = Tr\left[D^+(\lambda) D(-\lambda')\right]$$

$$= \left\langle \frac{D(\lambda)}{\sqrt{\pi}} \middle| \frac{D(-\lambda')}{\sqrt{\pi}} \right\rangle \pi \qquad = \pi\,\delta(\lambda - (-\lambda')) = \pi\,\delta(\lambda + \lambda')$$

prod scalare di H-S

$|D(\lambda)\rangle$ e' base ortonor. per lo sp operatori

$$= \int \frac{d^2\alpha \, d^2\lambda \, d^2\alpha'}{\pi} \, \underbrace{e^{\lambda(\alpha^* - \alpha'^*) - \lambda^*(\alpha - \alpha')}}_{\pi\,\delta(\alpha - \alpha')} \qquad W(\alpha) \, W(\alpha', 0)$$

$$= \pi \int d^2\alpha \; W(\alpha) \; W(\alpha, 0)$$

Funzioni di operatori in termini di operatori differenziali sulla Wigner

operatore differenziale

$$\mathcal{M}[\rho] \qquad W(\alpha, s, \mathcal{M}[\rho]) = D_s(\mathcal{M}) \, W(\alpha, s)$$

superoperatore → fn lineare di operatori

le mappe operatoriali in ottica quantistica sono composizioni di 4 mappe

$$a\,\cdot \qquad a^+\cdot$$
$$\cdot\, a \qquad \cdot\, a^+$$

esempio $\mathcal{M}[\rho] = a^\dagger \rho a = a^\dagger \cdot [\rho a] = a^\dagger \cdot [\cdot a [\rho]]$

$$\mathcal{M}[\rho] = a^\dagger a \rho = a^\dagger \cdot [a\rho] = a^\dagger \cdot [a \cdot [\rho]]$$

$$\boxed{D_s(a\,\cdot) = \alpha + \frac{1-s}{2}\frac{\partial}{\partial \alpha^*}}$$

derivata risp α^* dove considero α, α^* come indip

$$W(\alpha, s, a\rho) = \left(\alpha + \frac{1-s}{2}\frac{\partial}{\partial \alpha^*}\right) W(\alpha, s)$$

SERVE per trasformare equazioni operatoriali in equazioni differenziali con incognita $W(\alpha, s)$

$$W(\alpha, s, (a-\alpha)O) \overset{def}{=} \int \frac{d^2\lambda}{\pi^2} e^{\alpha\lambda^* - \alpha^*\lambda + \frac{s}{2}|\lambda|^2} \mathrm{Tr}\left[(a-\alpha)O \times e^{\lambda a^\dagger - \lambda^* a}\right] =$$

$$e^{\lambda a^\dagger - \lambda^* a} = e^{\lambda a^\dagger} e^{-\lambda^* a} e^{-\frac{|\lambda|^2}{2}}$$

$$= \int \frac{d^2\lambda}{\pi^2} e^{|\lambda|^2\left(\frac{s}{2} - \frac{1}{2}\right)} \mathrm{Tr}\left[\cancel{e^{\lambda(a^\dagger - \alpha^*)}} (a-\alpha) O\, \cancel{e^{-\lambda^*(a-\alpha)}}\right]$$

$$e^{-\lambda^*(a-\alpha)} \qquad e^{\lambda(a^\dagger - \alpha^*)}$$

$$= \int \frac{d\lambda\, d\lambda^*}{2\pi^2} e^{\frac{|\lambda|^2}{2}(s-1)} \mathrm{Tr}\left[-\frac{\partial}{\partial \lambda^*} e^{-\lambda^*(a-\alpha)}\, O\, e^{\lambda(a^\dagger - \alpha^*)}\right] =$$

$|\lambda|^2 = \lambda\lambda^*$

λ, λ^* trattati come var indip

$$\frac{\partial f(\lambda^*)}{\partial \lambda^*}$$

$$d^2\lambda = dx\,dy = du\,dv \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = du\,dv \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix} \right|$$

$\lambda = x + iy \qquad u = x + iy \qquad v = x - iy \qquad x = \frac{u+v}{2} \qquad y = -i\,\frac{u-v}{2}$

$$= \frac{du\,dv}{2} = \frac{d\lambda\, d\lambda^*}{2}$$

$$= \int \frac{d\lambda\, d\lambda^*}{2\pi^2} \mathrm{Tr}\left[e^{-\lambda^*(a-\alpha)}\, O\, e^{\lambda(a^+-\alpha^*)} \right] \underbrace{\frac{\partial}{\partial \lambda^*} e^{\frac{\lambda\lambda^*}{2}(s-1)}}_{e^{\frac{|\lambda|^2}{2}(s-1)}\frac{\lambda}{2}(s-1)}$$

integro per parti rispetto λ^*

$$= \int \frac{d\lambda\, d\lambda^*}{2\pi^2} \mathrm{Tr}\left[e^{-\lambda^*(a-\alpha)}\, O\, e^{\lambda a^+} \right] \underbrace{e^{-\lambda\alpha^*}\lambda}_{-\frac{\partial}{\partial\alpha^*} e^{-\lambda\alpha^*}} \frac{s-1}{2} e^{\frac{|\lambda|^2}{2}(s-1)}$$

$$= \frac{s-1}{2}\left(-\frac{\partial}{\partial\alpha^*}\right) \underbrace{\int \frac{d\lambda\, d\lambda^*}{2\pi^2} \mathrm{Tr}\left[e^{-\lambda^*(a-\alpha)}\, O\, e^{\lambda(a^+-\alpha^*)} \right] e^{\frac{|\lambda|^2}{2}}}_{W(\alpha, s, \hat{O})}$$

$$= \boxed{\frac{1-s}{2}\frac{\partial}{\partial\alpha^*} W(\alpha, s, O) = W(\alpha, s, (a-\alpha)O)}$$