OTTICA Q 11/11/20

isomorfismo tra su(2) complessa e sl(2 ℂ)

$$e^{\alpha\sigma_+ - \alpha^*\sigma_-} \overset{\text{IMP}}{=} \underbrace{e^{\xi\sigma_+}\, e^{\beta\frac{\sigma_z}{2}}\, e^{\gamma\sigma_-}}_{} \qquad \xi,\beta,\gamma\in\mathbb{C}$$

t. Lie per sl(2 ℂ)

$$e^{\alpha\sigma_+ - \alpha^*\sigma_-} = \mathbb{1}\cos|\alpha| + \frac{(\alpha\sigma_+ - \alpha^*\sigma_-)}{|\alpha|}\sin|\alpha|$$

$$e^{\frac{\beta}{2}\sigma_z} = \sum_n \frac{\left(\frac{\beta}{2}\right)^{2n}}{(2n)!}\underbrace{(\sigma_z)^{2n}}_{=\mathbb{1}} + \frac{\left(\frac{\beta}{2}\right)^{2n+1}}{(2n+1)!}\underbrace{(\sigma_z)^{2n+1}}_{=\sigma_z} = \underbrace{\mathbb{1}\,ch\frac{\beta}{2} + \sigma_z\, sh\frac{\beta}{2}}$$

$$\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \sigma_z^2 = \mathbb{1}$$

$$e^{\xi\sigma_+} = \mathbb{1} + \xi\sigma_+ \qquad e^{\gamma\sigma_-} = \mathbb{1} + \gamma\sigma_-$$

$$\sigma_+ = \begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad \sigma_+^2 = 0$$

$$\left(\mathbb{1}\, ch\frac{\beta}{2} + \sigma_z\, sh\frac{\beta}{2}\right)$$

$$(\mathbb{1}+\xi\sigma_+)\left(\cancel{\mathbb{1}\cos|\alpha| + \frac{\alpha\sigma_+ - \alpha^*\sigma_-}{|\alpha|}\sin|\alpha|}\right)(\mathbb{1}+\gamma\sigma_-) = \ldots$$

$$= \mathbb{1}\left(ch\frac{\beta}{2} + \frac{\xi\gamma}{2}e^{-\frac{\beta}{2}}\right) + \sigma_z\left(\gamma\xi\, e^{-\frac{\beta}{2}} + sh\frac{\beta}{2}\right) + e^{-\frac{\beta}{2}}\gamma\sigma_- + e^{-\frac{\beta}{2}}\sigma_+\xi$$

$$\overset{\text{IMP}}{=} \mathbb{1}\cos|\alpha| + \frac{\alpha\sigma_+ - \alpha^*\sigma_-}{|\alpha|}\sin|\alpha| \Rightarrow \begin{array}{l}\beta = \beta(\alpha)\\ \gamma = \gamma(\alpha)\\ \xi = \xi(\alpha)\end{array}$$

$$\mathbb{1} \to ch\frac{\beta}{2} + \frac{\xi\gamma}{2}e^{-\frac{\beta}{2}} = \cos|\alpha|$$

$$\sigma_z \to \gamma\xi\, e^{-\frac{\beta}{2}} + sh\frac{\beta}{2} = 0$$

$$\sigma_- \to \gamma e^{-\frac{\beta}{2}} = -\sin|\alpha|\frac{\alpha^*}{|\alpha|}$$

$$\ldots \Rightarrow \begin{array}{l}\xi = \frac{\alpha}{|\alpha|}tg|\alpha| = -\gamma^*\\ \beta = -2\ln\cos|\alpha|\end{array}$$

$$\sigma_+ \to \xi = -\gamma^*$$

NON c'è $\frac{1}{2}$!

$$e^{\alpha J_+ - \alpha^* J_-} \overset{*}{=} e^{\xi J_+}\, e^{\beta J_z}\, e^{-\xi^* J_-}$$

$$= e^{-\xi^* J_-}\, e^{-\beta J_z}\, e^{\xi J_+}$$

cambio algebra $J_+' \overset{def}{=} J_-$, $J_-' \overset{def}{=} J_+$, $J_z' \overset{def}{=} -J_z$

J'_+, J'_-, J'_z sono un'algebra di $su(2)$ → stesse cost di struttura

usando questa algebra ottengo

BCH $su(1,1)$

$$e^{\alpha K_+ - \alpha^* K_-} = e^{\xi K_+} e^{\beta K_z} e^{-\xi^* K_-} = e^{-\xi^* K_-} e^{-\beta K_z} e^{\xi K_+}$$

$$\xi = \frac{\alpha}{|\alpha|} \operatorname{th}|\alpha| \qquad \beta = -2 \ln \operatorname{ch}|\alpha|$$

↳ gli op iK_+, iK_-, K_z sono op di algebra di $su(2)$

$$[iK_+, iK_-] = -[K_+, K_-] = +2K_z$$

$$[K_z, iK_\pm] = \pm i K_\pm$$

uso di nuovo l'isomorfismo $su(2)$ compl e $sl(2\mathbb{C})$

⇒ stessa derivazione ma con gli op

$$e^{i\alpha \sigma_+ - i\alpha^* \sigma_-} = e^{i\xi \sigma_+} e^{\frac{\beta}{2}\sigma_z} e^{-i\gamma \sigma_-}$$

STATI QUANTISTICI

① Stato COERENTE → "stato classico" campo elm con fase e ampiezza "fissate"

↑ scoperti da Schroedinger

puntatore laser → st coerente

def $|\alpha\rangle$ $\alpha \in \mathbb{C}$ è coerente $\overset{def}{\Longleftrightarrow}$ $a|\alpha\rangle = \alpha|\alpha\rangle$

↑ lettere greche

autost di a

op distruz non è hermitiano $a^+ \neq a$

a val non è reale

PROPRIETÀ

(a) espansione su base numero

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \sum_n \psi_n |n\rangle$$

$$a|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} \underbrace{a|n\rangle}_{\sqrt{n}\,|n-1\rangle} = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{\sqrt{n}} |n\rangle = \alpha |\alpha\rangle$$

(b) $\langle\alpha|\alpha\rangle = 1$

$$\langle\alpha|\alpha\rangle = \left(e^{-\frac{|\alpha|^2}{2}}\right)^2 \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \underbrace{\langle m|n\rangle}_{\delta_{nm}} =$$

$$= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2} e^{|\alpha|^2} = 1$$

(c) Prob di avere n fotoni $\quad a^\dagger a = N = \sum_n n |n\rangle\langle n|$

$$P(n\,|\,|\alpha\rangle) = |\langle\alpha|n\rangle|^2 = \left| e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!}$$

regola di Born $\quad |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle$

Prob Poissoniana $\quad \langle N\rangle = \sum_n n\,P(n) = |\alpha|^2$

varianza $\quad \Delta N^2 = |\alpha|^2$

ha componenti su $\forall n$

se $\alpha = 0 \Rightarrow |\alpha\rangle = |0\rangle \leftarrow$ vuoto st di Fock $|0\rangle$ è un coerente $\quad \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \underset{\alpha=0}{=} |0\rangle$

(d) st coerenti NON sono ortogonali fra loro

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha^*\beta}$$

$$|\langle\alpha|\beta\rangle|^2 \stackrel{*}{=} e^{-|\alpha-\beta|^2} \qquad |\alpha| >> |\beta|$$

overlap decresce esponenzialmente

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \sum_{n,m=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \langle n| \frac{\beta^m}{\sqrt{m!}} |m\rangle \quad \delta_{nm} =$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n!} = e^{-\left(\frac{|\alpha|^2}{2}+\frac{|\beta|^2}{2}\right)+\alpha^*\beta}$$

(e) Displacement $\quad D(\alpha) \stackrel{def}{=} e^{\alpha a^+ - \alpha^* a}$ $\quad = e^{\alpha a^+} e^{-\alpha^* a} e^{-\frac{|\alpha|^2}{2}}$ $\quad [a, a^+] = 1$

unitario

BCH wh

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

$$= e^{-\alpha^* a} e^{\alpha a^+} e^{+\frac{|\alpha|^2}{2}}$$

$$D(\alpha)|0\rangle = |\alpha\rangle$$

$$D(\alpha)|0\rangle = e^{\alpha a^+} e^{-\alpha^* a} e^{-\frac{|\alpha|^2}{2}} |0\rangle = e^{\alpha a^+} |0\rangle e^{-\frac{|\alpha|^2}{2}}$$

$$\alpha^* a \overset{\shortparallel}{} 0$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (a^{+n}|0\rangle) e^{-\frac{|\alpha|^2}{2}} = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle = |\alpha\rangle$$

$$a^{+n}|0\rangle \to |n\rangle\sqrt{n!}$$

(f) Coerente spostato $\quad D(\alpha)|\beta\rangle = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} |\alpha+\beta\rangle$

$$D(\alpha)|\beta\rangle = e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha a^\dagger}\, e^{-\alpha^* \overset{\beta}{a}}\, |\beta\rangle =$$

$$= e^{-\frac{|\alpha|^2}{2}-\alpha^*\beta}\, e^{\alpha a^\dagger} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}}|n\rangle\, e^{-\frac{|\beta|^2}{2}} =$$

$$= e^{\cdots} \sum_{n,m=0}^{\infty} \frac{\alpha^m}{m!}\, a^{\dagger m}\, \frac{\beta^n}{\sqrt{n!}}\, |n\rangle = e^{\cdots} \sum_{n,m} \frac{\alpha^m \beta^n}{m!\, n!} \sqrt{(n+m)!}\, |n+m\rangle$$

$$= \frac{\sqrt{(n+m)!}}{\sqrt{\cancel{n!}}}\, |n+m\rangle$$

$$\sqrt{n!}\,|n\rangle$$

$$|n+m\rangle = \frac{(a^\dagger)^{n+m}}{\sqrt{(n+m)!}}|0\rangle = \frac{a^{\dagger m}\, a^{\dagger n}|0\rangle}{\sqrt{(n+m)!}}$$

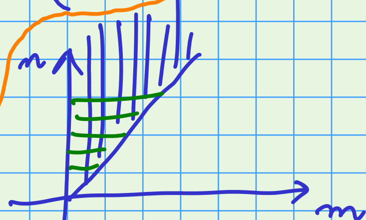
$$= \frac{a^{\dagger m}\sqrt{n!}}{\sqrt{(n+m)!}}\, |n\rangle \qquad a^{\dagger m}|n\rangle = \sqrt{\frac{(n+m)!}{n!}}\, |n+m\rangle$$

$m' = n+m$

$$= e^{--} \sum_{n=0}^{\infty} \sum_{m'=n}^{\infty} \frac{\alpha^{m'-n}\, \beta^n}{(m'-n)!\, n!} \sqrt{m'!}\, |m'\rangle =$$

$$= e^{--} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \binom{m}{n} \frac{1}{\sqrt{m!}}\, \alpha^{m-n}\, \beta^n\, |m\rangle$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m}$$



$$= (\alpha+\beta)^m$$

$$(x+y)^k = \sum_{n=0}^{k} \binom{k}{n} x^n y^{k-n}$$

$$= e^{\cdots} \sum_{m=0}^{\infty} \frac{(\alpha+\beta)^m}{\sqrt{m!}}\, |m\rangle \propto |\alpha+\beta\rangle$$

9) coerenti → stati a minima indet → soddisfano relaz indet con l'uguale

$\Delta p\, \Delta q \geq \frac{1}{2}$ coerenti $\Delta p\, \Delta q = \frac{1}{2}$

$$q = \sqrt{\frac{\hbar}{2\omega m}}\,(a + a^+) \qquad p = -i\sqrt{\frac{\hbar\omega m}{2}}\,(a - a^+)$$

def di a, a^+ dall'op posiz dell'osc arm

$$\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2$$

$a|\alpha\rangle = \alpha|\alpha\rangle$

$\langle\alpha|a^+ = \alpha^*\langle\alpha|$

$\alpha^*\alpha = |\alpha|^2$

$$\langle q^2\rangle = \frac{\hbar}{2\omega m}\langle\alpha|a^2 + a^{+2} + \underbrace{aa^+}_{= a^+a+1} + \underbrace{a^+a}_{|\alpha|^2}|\alpha\rangle =$$

α^2, α^{*2}

$$\frac{\hbar}{2\omega m}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1\right)$$

$$\langle q\rangle^2 = \left(\sqrt{\frac{\hbar}{2\omega m}}\,\langle\alpha|a + a^+|\alpha\rangle\right)^2 = \left(\sqrt{\frac{\hbar}{2\omega m}}\,(\alpha + \alpha^*)\right)^2 =$$

α, α^*

$$= \frac{\hbar}{2\omega m}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right)$$

$$\left.\begin{aligned}\Delta q^2 &= \langle q^2\rangle - \langle q\rangle^2 = \frac{\hbar}{2\omega m}\\ \Delta p^2 &= \frac{\hbar\omega m}{2}\end{aligned}\right\} \quad \Delta p^2\Delta q^2 = \frac{\hbar^2}{4}$$

$$\Delta p\,\Delta q = \frac{\hbar}{2}$$

Coerente di Schroedinger $\rightarrow$ oscillatore armonico

$q \rightarrow$ posiz dell'osc

$p \rightarrow$ momento

radiazione q, p <u>NON</u> sono posizione e momento del fotone

$q \rightarrow$ "posizione" dell'osc che rappresenta il modo

$$p \propto (a - a^+) \propto -\frac{\partial\vec{A}}{\partial t} = \vec{E}$$

p è l'osservabile campo elettrico

generalizzazione di q, p per la radiazione

→ l'op quadratura def $\underbrace{a_\varphi \overset{def}{=} \frac{1}{\sqrt{2}}\left(a^+ e^{i\varphi} + a e^{-i\varphi}\right)}$

$q = a_{\varphi=0}$

$p = a_{\varphi=\frac{\pi}{2}}$

↳ è osservabile, op autoaggiunto

$a_\varphi^+ = a_\varphi$

↳ misurato dall'apparato omodina

attenzione: a volte $\frac{1}{2}$

quadrature che differiscono per $\frac{\pi}{2}$ sono coniugate

$$\left[a_\varphi, a_{\varphi+\frac{\pi}{2}}\right] = \frac{1}{2}\left[a^+e^{i\varphi}+ae^{-i\varphi}, \underbrace{ia^+e^{i\varphi}-iae^{-i\varphi}}_{a_{\varphi+\frac{\pi}{2}}}\right] =$$

$$= i = [q, p]$$

$[a, a^+] = 1$

$|\alpha\rangle$ è a minima indet $\Delta a_\varphi \, \Delta a_{\varphi+\frac{\pi}{2}} = \frac{1}{2}$

Th COMPLETEZZA degli st coerenti

over completezza

$|x\rangle$ base $\Leftrightarrow \int dx\, |x\rangle\langle x| = \mathbb{1}$

$$\boxed{\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \mathbb{1}}$$

NOTAZIONE

$\alpha = \rho e^{i\varphi}$

$$\int_{\mathbb{C}} d^2\alpha \overset{def}{=} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy = \int_0^{2\pi} d\varphi \int_0^{+\infty} \rho\, d\rho$$

$x + iy$

$$= \int d\alpha \, d\alpha^*$$

α, α^* considerate come var indip

$$= \int d^2\alpha \int d^2\beta \; \delta^{(2)}(\alpha^* - \beta)$$