OTTICA Q 10/11/20

METODI ALGEBRICI

$$e^{xX} A e^{-xX} = A + x[X,A] + \dots \frac{x^n}{n!} \underbrace{[X,[X \;\; [X,A]]]}_{n \text{ volte}} + \dots$$

$$= \sum_{n=0} \frac{x^n}{n!} \underbrace{[\;\;}_{n} [X,A]] \overset{def}{=} \exp[x \, \mathrm{ad} X] A$$

$$\mathrm{ad}(X) A \overset{def}{=} [X,A]$$

$$e^{xX} A e^{-xX} \overset{def}{=} A(x) \qquad \frac{dA(x)}{dx} = X \underbrace{e^{xX} A e^{-xX}}_{A(x)} - \underbrace{e^{xX} A e^{-xX}}_{A(x)} X$$

$$= [X, A(x)] = \mathrm{ad}(X)\, A(x)$$

eq differ $\frac{dA(x)}{dx} = \mathrm{ad}(X) A(x)$

condiz init $A(x=0) = A$

$$\frac{df}{dx} = \alpha f(x) \Rightarrow f(x) = f_0 e^{\alpha x}$$

$$\overset{e^{xX} A e^{-xX}}{A}(x) = \exp[x \underbrace{\mathrm{ad}(X)}_{\alpha}] A$$

formula utile solo quando gli op A, X formano una algebra

$$e^{\alpha G_z} G_+ e^{-\alpha G_z} = G_+ e^{\alpha/2} \quad \text{esempio}$$

$$X = \alpha a^+ - \alpha^* a \qquad A = a$$

op displacement $D(\alpha) \overset{def}{=} e^{\alpha a^+ - \alpha^* a}$

$$e^{\alpha a^+ - \alpha^* a} \, a \, e^{-(\alpha a^+ - \alpha^* a)} = D(\alpha)\, a\, D^+(\alpha) =$$

$$a + [\alpha a^+ - \alpha^* a, a] + \frac{1}{2!}\left[\alpha a^+ - \alpha^* a, [-\alpha]\right] + \dots$$

$= a - \alpha$

LEMMA (facoltativo)

$X(t) \in$ algebra di Lie $\Rightarrow e^{-X(t)} \frac{d}{dt} e^{X(t)} = \varphi\left[-ad(X)\right]\frac{dX}{dt}$

$\varphi(x) \stackrel{def}{=} \frac{e^{x}-1}{x}$

$G(s,t) \stackrel{def}{=} e^{sX(t)} \qquad s \in \mathbb{R}$

$A(\cancel{s},t) \stackrel{def}{=} G^{-1}\frac{\partial G}{\partial s} = e^{-sX}\underbrace{\frac{\partial e^{sX}}{\partial s}}_{Xe^{sX}} = e^{-sX} X e^{sX} = X(t)$

$B(s,t) \stackrel{def}{=} G^{-1}\frac{\partial}{\partial t} G = e^{-sX}\frac{\partial e^{sX}}{\partial t}$

$[A,B] \stackrel{def}{=} G^{-1}\frac{\partial G}{\partial s} G^{-1}\frac{\partial G}{\partial t} - G^{-1}\frac{\partial G}{\partial t} G^{-1}\frac{\partial G}{\partial s} =$

$G^{-1}\frac{\partial G}{\partial s}G^{-1} = -\frac{\partial G^{-1}}{\partial s}, \qquad G^{-1}\frac{\partial G}{\partial t}G^{-1} = -\frac{\partial G^{-1}}{\partial t}$

$\frac{\partial G^{-1}}{\partial x} = \frac{\partial}{\partial x}\left(G^{-1} G G^{-1}\right) = \frac{\partial G^{-1}}{\partial x}\mathbb{1} + G^{-1}\frac{\partial G}{\partial x}G^{-1} + \frac{\partial G^{-1}}{\partial x}$

$G^{-1}\frac{\partial G}{\partial x}G^{-1} = -\frac{\partial G^{-1}}{\partial x}$

$= -\frac{\partial G^{-1}}{\partial s}\frac{\partial G}{\partial t} + \frac{\partial G^{-1}}{\partial t}\frac{\partial G}{\partial s}$

$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t}\left(G^{-1}\frac{\partial G}{\partial s}\right) = \frac{\partial G^{-1}}{\partial t}\frac{\partial G}{\partial s} + G^{-1}\frac{\partial^2 G}{\partial t \partial s}$

$\frac{\partial B}{\partial s} = \frac{\partial G^{-1}}{\partial s}\frac{\partial G}{\partial t} + G^{-1}\frac{\partial^2 G}{\partial t \partial s}$

$\Rightarrow \frac{\partial A}{\partial t} - \frac{\partial B}{\partial s} = \frac{\partial G^{-1}}{\partial t}\frac{\partial G}{\partial s} - \frac{\partial G^{-1}}{\partial s}\frac{\partial G}{\partial t}$

$[A,B] = \frac{\partial A}{\partial t} - \frac{\partial B}{\partial s}$ eq differ in termini di B:

$\boxed{\frac{\partial B}{\partial s} = \underset{\frac{\partial A}{\partial t}}{\dot{A}} - \underbrace{ad(A)B}_{[A,B]}}$

$\frac{\partial f(s)}{\partial s} = \alpha - \beta f(s)$

ha soluz $\boxed{f(s) = \frac{1-e^{-s\beta}}{\beta}\alpha} \Leftarrow \frac{\partial f}{\partial s} = \underbrace{e^{-s\beta} - \alpha}_{-\beta f} + \alpha$

$$B(s) = \frac{1-e^{-s\,ad(A)}}{ad(A)}\frac{\partial A}{\partial t} = s\underbrace{\frac{e^{-s\,ad(A)}-1}{-s\,ad(A)}}_{s\varphi[-s\,ad(A)]}\frac{\partial A}{\partial t} =$$

β α

$$= s\varphi[-s\,ad(A)]\frac{\partial A}{\partial t}$$

LEMMA

tesi: $s=1$ $\quad B(s=1) = \boxed{e^{X(t)}\frac{\partial}{\partial t}e^{X(t)} = \varphi[-ad(A)]\frac{\partial A}{\partial t}}$

facoltativo

Th BCH $\to$ Baker-Campbell-Hausdorff

$$X, Y \in a \Rightarrow \boxed{\ln(e^X e^Y) = X + \int_0^1 dt\, \psi(e^{adX}e^{t\,adY})\,Y}$$

$\uparrow$ formula BCH

$$\psi(x) \stackrel{def}{=} \frac{x\ln x}{x-1}$$

parametro $\mathbb{R}$

$$L(t) \stackrel{def}{=} \ln(e^X e^{tY}) \qquad e^{\ln O} = O$$

$\forall N \in a$

$$\underbrace{e^{L(t)}Ne^{-L(t)}}_{\| \atop e^{ad(L)}N} = e^X \underbrace{e^{tY}Ne^{-tY}}e^{-X}$$

$$= \underbrace{e^X e^{t\,ad(Y)}N e^{-X}}_{e^{ad(X)}e^{t\,ad(Y)}N}$$

$\forall N$

$$e^{ad(L)} = e^{ad(X)} e^{t\,ad(Y)}$$

$$\varphi(-\ln z) \underset{\text{def di } \varphi}{=} \frac{e^{-\ln z} - 1}{-\ln z} = \frac{z^{-1} - 1}{-\ln z} = \frac{1 - z^{-1}}{\ln z} = \frac{z - 1}{z \ln z}$$

def ψ

$$\overset{\downarrow}{=} \frac{1}{\psi(z)}$$

$$z = e^{ad X} e^{t\,ad Y} = e^{ad L} \Rightarrow -\ln z = -ad_L$$

$$\psi(z)\,\varphi(-\ln z) = 1 \Rightarrow \boxed{\psi\left(e^{ad X} e^{t\,ad Y}\right) \varphi(-ad L) = 1\!\!1}$$

$$\frac{dL(t)}{dt} = 1\!\!1 \frac{dL}{dt} = \psi\left(e^{ad X} e^{t\,ad Y}\right) \underbrace{\varphi(-ad L) \frac{dL}{dt}}_{= Y}$$

$$\frac{d e^L}{dt} \underset{\text{def } L}{=} \frac{d}{dt}\left(e^X e^{tY}\right) = e^X e^{tY} Y = e^L Y$$

$L = \ln\left(e^X e^{tY}\right)$

$$e^{-L} \frac{d e^L}{dt} = Y$$

$$\overset{\text{Lemma}}{=} \varphi[-ad L] \frac{dL}{dt}$$

$$\int_0^1 dt \frac{dL}{dt} = \int_0^1 dt\, \psi\left(e^{ad X} e^{t\,ad Y}\right) Y$$

$$L(t) = \ln\left(e^X e^{tY}\right)$$

$$L(t=1) - L(t=0) = \int_0^1 dt\, \psi\left(e^{ad X} e^{t\,ad Y}\right) Y$$

$L(t=1) = \ln(e^X e^Y)$, $L(t=0) = \ln e^X = X$

$$\ln\left(e^X e^Y\right) = X + \int_0^1 dt\, \psi\left(e^{ad X} e^{t\,ad Y}\right) Y$$

SS di $\Psi(x) \stackrel{def}{=} \frac{x \ln x}{x-1} = \frac{x}{x-1} \ln(x+1-1) = \frac{x}{\cancel{x-1}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^{\cancel{n}}}{n}$

$\ln(x+1) = \sum_{n=1}^{\infty} \frac{x^n (-1)^{n+1}}{n}$

$$= x \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(1-x)^n}{n+1} x = \Psi(x)}$$

dim del th di Lie

$\underbrace{\forall X, Y \in a \Rightarrow \exists Z(X,Y) \in a \text{ t.c. } e^X e^Y = e^Z}$ INTERNA

$(\Leftrightarrow)$ l'esp di un'algebra è un gruppo

- ASSOCIATIVA → prod op
- $\exists$ el neutro → $\mathbb{1} = e^0$ vett nullo dell'algebra
- $\exists$ inverso

banali

$X \in a$ l'inverso di e^X è e^{-X}

uso BCH

$Z = X + \int_0^1 dt \, \Psi(e^{adX} e^{t\,adY}) Y \in a$ perché contiene solo $X, Y \in a$, e i commutatori

$e^X e^Y = e^Z$ formula di BCH

"specie" di viceversa

$X_1 \dots X_n$

$\underbrace{G \text{ gruppo di Lie}, \quad D(g) \text{ è una rappres di } G, \quad g \in G \quad g = g(a_1 \dots a_n) \quad \text{t.c. } \mathbb{1} = g(a_1 = 0 \dots a_n = 0)}$

$\Rightarrow X_k \stackrel{def}{=} \left. \frac{\partial D(g)}{\partial a_k} \right|_{a_1 \dots a_n = 0}$

formano un'algebra di Lie

no dimostrazione

Formula di BCH per operatori con commutatore commutante

A, B operat. t.c. $[A,B]$ commuta con A, B

$$\big[[A,B],A\big]=\big[[A,B],B\big]=0 \Rightarrow e^A e^B \overset{\downarrow}{=} e^{A+B+\frac{1}{2}[A,B]} \overset{\uparrow}{=} e^{A+B}\, e^{+\frac{1}{2}[A,B]}$$

commuta

$$\Gamma(t) \overset{\text{def}}{=} \underbrace{e^{\operatorname{ad}A}\, e^{t\operatorname{ad}B}}$$

$$\Gamma B = e^{\operatorname{ad}A}\, \cancel{e^{t\operatorname{ad}B}}\, B = e^{\operatorname{ad}A} B =$$

$$= \sum_{n=0}^{\infty} \frac{[A,[A,\dots[A,B]]]}{n!} = B + [A,B] + \frac{1}{2!}\underbrace{[A,[A,B]]}_{\text{''0 + ip''}} + 0$$

$$\Gamma[A,B] = \sum_n \frac{[A,[A,\dots[A,B]]]}{n!} \overset{n=0}{=} [A,B] + 0$$

$$(1\!\!1-\Gamma)^n B = \begin{cases} n=0 & B \\ n=1 & (1\!\!1-\Gamma)B = B - B - [A,B] = -[A,B] \\ n>1 & 0 \end{cases}$$

ss di ψ

$$\psi(\Gamma)B \overset{\downarrow}{=} \underbrace{\Gamma \sum_{n=0}^{\infty} \frac{(1-\Gamma)^n}{n+1}}_{\psi(\Gamma)} B \overset{n=0,1}{=} \Gamma\Big(\overset{n=0}{B} - \frac{\overset{n=1}{[A,B]}}{2}\Big) =$$

$$= \overbrace{B + [A,B]}^{\Gamma B} - \overbrace{\frac{[A,B]}{2}}^{\Gamma\left(-\frac{[A,B]}{2}\right)} = B + \frac{1}{2}[A,B]$$

Uso Formula BCH

$$Z = A + \int_0^1 dt\, \psi\big(\overbrace{e^{\operatorname{ad}A}\, e^{t\operatorname{ad}B}}^{\Gamma(t)}\big) B = A + \int_0^1 dt \Big(B + \frac{1}{2}[A,B]\Big)$$

$$= A + B - \frac{1}{2}[A,B]$$

$$e^A e^B = e^Z = e^{A+B+\frac{1}{2}[A,B]}$$

caso particolare

$[A,B] \in \mathbb{C}$

$[A,[A,B]]=0$

ottica quantistica → wh

$[a, a^\dagger] = 1$

BCH mom angolare: algebra di su(2): (facoltativo)

$$e^{\alpha J_+ - \alpha^* J_-} = e^{\xi J_+} e^{\beta J_z} e^{-\xi^* J_-} = e^{-\xi^* J_-} e^{-\beta J_z} e^{\xi J_+}$$

$\alpha \in \mathbb{C}$ $\quad \xi = \frac{\alpha}{|\alpha|} \operatorname{tg}|\alpha|$ $\quad \beta = -2 \ln \cos|\alpha|$

$$e^{\beta J_z} = (\cos|\alpha|)^{-2 J_z}$$

l'algebra reale di su(2) $\xrightarrow{\text{esponenziaz}}$ Gruppo SU(2)

l'algebra complessa di su(2): isomorfa all'algebra di sl(2, $\mathbb{C}$)

sl(2 $\mathbb{C}$) ← algebra mat 2×2 a traccia nulla →

$\{\sigma_x, \sigma_y, \sigma_z\}$ ← base

usiamo il t. di Lie per sl(2, $\mathbb{C}$) → usiamo isomorfismo per dire che tutte le relazioni ottenute con matrici 2×2 valgono genericam per operatori generici con algebra di su(2)