OTTICA Q 3/11/20

$\vec{A} = \sum_{k\sigma} \sqrt{\frac{\hbar}{\dots\omega_k}} \; - - -$

INTERAZ RADIAZ MATERIA:

$$H_i = -q\vec{r}\cdot\vec{E} = q\sum_{k\sigma}\sqrt{\frac{\hbar\boxed{\omega_k}}{2\varepsilon_0 L^3}}\; q\; \boxed{\vec{r}}\cdot\vec{e}_{k\sigma}\left(\boxed{-i}\, a_{k\sigma}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + h.c.\right)$$

$\uparrow\; \vec{E} = -\frac{\partial\vec{A}}{\partial t}$

$\sum_j |e_j\rangle\langle e_j| = \mathbb{1}$ $\qquad \sum_k |e_k\rangle\langle e_k| = \mathbb{1}$

$|e_m\rangle$ autost dell'$H_{at} = \sum_j e_j |e_j\rangle\langle e_j|$

$\vec{d}_{00} = \vec{d}_{11} = 0 \leftarrow$ simmetria potenziale centrale $\Rightarrow U(\vec{r}) = U(-\vec{r})$
$\Rightarrow$ anche gli autost sono pari, ma $\vec{r}$ è dispari

$\vec{r} \to -\vec{r}$

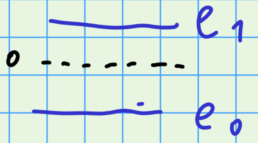
$\langle e_0|\vec{r}|e_0\rangle = 0$

$\langle e_1|\vec{r}|e_1\rangle = 0$

$d_{10} = d_{01} \neq 0$

$$\vec{d}_{\underset{\text{discreti}}{jk}} \overset{def}{=} \underset{\shortparallel\, -e}{q} \langle e_j|\underset{\uparrow\text{ continuo}}{\vec{r}}|e_k\rangle \leftarrow \text{ampiezza di transz di dipolo}$$

ATOMO a 2 livelli



$\hbar\omega_0$ = differ di en

$$H_{at} = \frac{\hbar\omega_0}{2}\sigma_z \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

MATRICI PAULI:

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \overset{\begin{pmatrix}0&0\\1&0\end{pmatrix}}{|e_0\rangle\langle e_1|} + \boxed{\overset{\begin{pmatrix}0&1\\0&0\end{pmatrix}}{|e_1\rangle\langle e_0|}}$$

$$\underset{\uparrow\text{ autost del livello eccitato a val } \frac{\hbar\omega_0}{2}}{|e_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}}, \quad |e_0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \leftarrow \text{autost di livello ground con a val } -\frac{\hbar\omega_0}{2}$$

$$H_{at} = \frac{\hbar\omega_0}{2}\sigma_z$$

$$\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = -i\,|e_1\rangle\langle e_0| + i\,|e_0\rangle\langle e_1|$$

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \underset{\begin{pmatrix}1&0\\0&0\end{pmatrix}}{|e_1\rangle\langle e_1|} - \underset{\begin{pmatrix}0&0\\0&1\end{pmatrix}}{|e_0\rangle\langle e_0|}$$

$\sigma_-|e_0\rangle = 0$

$\sigma_-|e_1\rangle = |e_0\rangle$

$$\sigma_0 = \mathbb{1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\sigma_+ = |e_1\rangle\langle e_0| = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad \sigma_- = |e_0\rangle\langle e_1| = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$\sigma_j^2 = \mathbb{1}$ $\quad j = x, y, z, 0$

$$\sigma_+|e_0\rangle = |e_1\rangle\overbrace{\langle e_0|e_0\rangle} = |e_1\rangle$$

$$H_i = q \sum_{\vec{k}6} \sqrt{\cdots} \; \underbrace{\sum_{jm} |e_j\rangle\langle e_m| \, \vec{d}_{jm}\cdot\vec{e}_{\vec{k}6}} \left(-i a e^{\cdots} + h.c.\right)$$

$d_{00} = d_{11} = 0$ sopravvive $|e_0\rangle\langle e_1| = \sigma_-$

$|e_1\rangle\langle e_0| = \sigma_+$

$$= q \sum_{k} \sqrt{\frac{\cdots}{\cdots}} \, (\underbrace{\sigma_- + \sigma_+}) \, \vec{d}_{01}\cdot\vec{e}_{k6} \left(-i a e^{\cdots} \cancel{+h.c.}\right)$$

$$= \hbar \sum_{\vec{k}6} (\sigma_+ + \sigma_-)\left(g_{\vec{k}6} a_{k6} + g^*_{k6} a^\dagger_{\vec{k}6}\right) =$$

$$g_{\vec{k}6} \overset{def}{=} -i \, \vec{d}_{01}\cdot\vec{e}_{k6} \sqrt{\frac{\hbar\omega_k}{\varepsilon_0 2 L^3}} \, e^{i(\vec{k}\cdot\vec{r} - \cancel{\omega t})}$$

$\sigma_+ a$ ← distrugge fotone e eccita l'atomo } conservano energia

$\sigma_- a^\dagger$ crea " diseccita atomo }

$\sigma_+ a^\dagger$ ← " " eccita " } non conservano energia

$\sigma_- a$ ← distrugge " diseccita " }

RWA rotating wave approx → conserviamo solo i termini che conservano l'energia

$$\overset{RWA}{\simeq} \underbrace{\hbar \sum_{\vec{k}'6} \sigma_+ a_{k'} g_{k'6} + \sigma_- a^\dagger_{k'6} g^*_{k'6}}$$ ← somma corre solo sui k' t.c. $k' \sim \omega_0/c$

$$H_{TOT} = \underbrace{\sum_{\vec{k}6} \hbar\omega_k \left(a^\dagger_{\vec{k}6} a_{\vec{k}6} + \frac{1}{2}\right)}_{H_{rad}} + \underbrace{\frac{\hbar\omega_0}{2}\sigma_z}_{H_{at}} + \underbrace{\hbar \sum_{k'6} \sigma_+ a g + \sigma_- a^\dagger g^*}_{H_i}$$

Hamiltoniana di Jaynes-Cummings

GIUSTIFICAZ della RWA:

scriviamo H_i in pittura interazione $\quad H_1 = H_{rad} + H_{at}$

$$H_i^I = e^{\frac{i}{\hbar}H_1 t} \, H_i \, e^{-\frac{i}{\hbar}H_1 t} =$$

$$= U^+ \quad H_i \quad U$$

$$[H_{rad}, H_{at}] = 0 \Rightarrow e^{\frac{i}{\hbar}(H_{at}+H_{rad})t} =$$

$$= e^{\frac{i}{\hbar}H_{at}t} \, e^{\frac{i}{\hbar}H_{rad}t}$$

$$H_i^I = U^+(\sigma_+ + \sigma_-)\overset{UU^+}{\vee}(ga + g^* a^+)\,U =$$

$$= e^{\frac{i}{\hbar}H_{at}t}(\sigma_+ + \sigma_-)e^{-\frac{i}{\hbar}H_{at}t} \quad e^{+\frac{i}{\hbar}H_{rad}t}\,(ga + g^* a^+)\,e^{-\frac{i}{\hbar}H_{rad}t}$$

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \dots$$

$$+ \frac{\alpha^n}{n!}\underbrace{[A,[A,[A,}_{n \text{ volte}}\;[A,B]]]] + \dots$$

$$e^{\alpha A} B e^{-\alpha A} = \sum_{m=0}^{\infty} \frac{\alpha^n A^n}{n!}\, B\, \frac{(-\alpha)^m A^m}{m!} =$$

$$= (\mathbb{1} + \alpha A + \frac{\alpha^2 A^2}{2} + \dots)\, B\, (\mathbb{1} - \alpha A + \frac{\alpha^2 A^2}{2} - \dots) =$$

$$= \underset{\uparrow \text{ ordine } 0 \text{ in } \alpha}{B} + \alpha(\underbrace{AB - BA}_{[A,B]}) + \underbrace{\frac{\alpha^2}{2}(-2ABA + \overset{[A,[A,B]]}{A^2B} + BA^2)}_{\text{ord } 2 \text{ in } \alpha} + \dots$$

$\uparrow$ ord 1 in α

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \dots$$

$$e^{\frac{i\omega_0}{2}\sigma_z t}\,\sigma_+\,e^{-\frac{i\omega_0}{2}\sigma_z t} = \sigma_+ + \frac{i\omega_0 t}{\cancel{2}}(\cancel{2}\sigma_+) + \left(\frac{i\omega_0 t}{\cancel{2}}\right)^2 \frac{1}{2!}[\sigma_z, \underbrace{2\sigma_+]}_{\cancel{4}\sigma_+} + \dots$$

$$\boxed{[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm}$$

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} - \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} =$$

$$= \sigma_+\left(1 + \frac{i\omega_0 t}{\cancel{2}} + \left(\frac{i\omega_0 t}{\cancel{2}}\right)^2\frac{1}{2!} + \dots \left(\frac{i\omega_0 t}{\cancel{2}}\right)^n \frac{1}{n!} + \dots\right) =$$

$$= \sigma_+ e^{i\omega_0 t} = \sigma_+^I(t)$$

$$\sigma_-^I(t) = \sigma_- e^{-i\omega_0 t}$$

$$[a^+a, a] = [a^+,a]a + a^+\cancel{[a,a]}$$

$$= -a$$

$$a^I(t) = e^{i\omega a^+ a t + \frac{i\omega t}{2}}\, a\, e^{-i\omega a^+ a t - \frac{i\omega t}{2}}$$

$$[a_k, a^+_{k'}] = 0 \text{ se } \vec{k} \neq \vec{k}'$$

$$= a + i\omega t \underbrace{[a^+a, a]}_{-a} + \frac{(i\omega t)^2}{2!} \times$$

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha [A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \dots$$

$$\times \underbrace{[a^+a, (-a)]}_{+a} + \dots = a\left(1 - i\omega t + \frac{(i\omega t)^2}{2!} - \frac{(i\omega t)^3}{3!} + \dots\right)$$

$$= a e^{\overset{+}{-} i\omega t} = a^I$$

$$(a^I)^+ = a^+ e^{i\omega t}$$

$$H_i^I(t) = \sum_k \hbar \left(\sigma_+ e^{i\omega_0 t} + \sigma_- e^{-i\omega_0 t}\right)\left(g_k a_k e^{-i\omega_k t} + g_k^* a_k^+ e^{i\omega_k t}\right)$$

$$= \hbar \sum_k \Big(\sigma_+ a g e^{i(\omega_0 - \omega_k)t} + \sigma_- a^+ g^* e^{-i(\omega_0 - \omega_k)t} +$$

$$+ \sigma_+ a^+ g^* e^{i(\omega_0 + \omega_k)t} + \sigma_- a g e^{-i(\omega_0 + \omega_k)t}\Big)$$

ω e ω_0 sono freq ottiche $\sim 10^{15}$ Hz $\Rightarrow$ i termini con $\omega + \omega_0$ o i termini con $\omega_k - \omega_0$ dove ω_k è diverso da ω_0 sono termini che oscillano molto rapidamente

$\Rightarrow$ su scale di tempi $t \gg \frac{1}{\omega_k} \sim \frac{1}{\omega_0}$ i termini con $+$ si mediano a zero e anche i termini dove $\omega_0 \gg \omega_k$

$\omega_0 \ll \omega_k$

$$H_i^{I,RWA} = \hbar \sum_{k,b_k} \left(\sigma_+ a g e^{it\Delta} + \sigma_- a_g^{+*} e^{-it\Delta} \right)$$

$\Delta = \omega_0 - \omega_k$ e conserviamo solo i termini con Δ piccolo

$\uparrow$ detuning

$\Delta = 0$ cioè risonanza perfetta $H_i^{I,RWA}$ è indip da t

$\Rightarrow |\psi^I(t)\rangle = e^{-\frac{i}{\hbar} H_i^{I,RWA} t} |\psi^I(0)\rangle$ ∃ solo in questo caso

Cosa succede ad atomo a 2 livelli in interaz con la radiaz

lavoriamo con detuning 0 → un modo della radiaz risonante

① Caso in cui radiaz classica

② " " " " quantistica

① radiaz classica: passo indietro $a \to \alpha \in \mathbb{C}$

$a^+ \to \alpha^*$

$$H_i^{RWA} = \hbar g \left(\alpha \sigma_+ + \alpha^* \sigma_- \right)$$

$$|\psi^I(t)\rangle = e^{-\frac{i}{\hbar} H_i t} |\psi^I(0)\rangle$$

$$U = \sum_{m=0}^{\infty} \left(-\frac{it}{\hbar} \right)^m \frac{H_i^m}{m!} = \sum_{m=0}^{\infty} \frac{1}{m!} (-igt)^m \underbrace{\left(\alpha\sigma_+ + \alpha^*\sigma_- \right)^m}$$

$$\left(\alpha\sigma_+ + \alpha^*\sigma_- \right)^{2m} \underset{\uparrow}{=} \left(\underbrace{|e_0\rangle\langle e_0| + |e_1\rangle\langle e_1|}_{\mathbb{1}} \right) \overbrace{\alpha^m \alpha^{*m}}^{|\alpha|^{2m}}$$

$\sigma_+\sigma_+ \; \sigma_-\sigma_-$

$\sigma_+\sigma_+ = 0$ $\quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$

$\sigma_-\sigma_- = 0$

$\sigma_+ = |e_1\rangle\langle e_0|$

$\sigma_- = |e_0\rangle\langle e_1|$

$\sigma_+\sigma_+ = |e_1\rangle \overset{=0}{\langle e_0 | e_1\rangle} \langle e_0| = 0$

sopravvivono solo i termini $\sigma_+\sigma_-\sigma_+\sigma_- = |e_1\rangle \overset{=1}{\langle e_0|e_0\rangle} \overset{=1}{\langle e_1| \dots} \overset{''1}{|e_0\rangle}\langle e_1|$

$\sigma_-\sigma_+\sigma_-\sigma_+$

$$\sigma_+\sigma_-\sigma_+\sigma_- = |e_1\rangle\langle e_1| = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\sigma_-\sigma_+\sigma_-\sigma_+ = |e_0\rangle\langle e_0| = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\boxed{\begin{pmatrix}0&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1\!\!1}$$

$$\boxed{(\alpha\sigma_+ + \alpha^*\sigma_-)^{2m} = |\alpha|^{2m}\, 1\!\!1}$$

$$(\alpha\sigma_+ + \alpha^*\sigma_-)^{2m+1} = (\qquad)^{2m}(\alpha\sigma_+ + \alpha^*\sigma_-) =$$

$$= |\alpha|^{2m}(\alpha\sigma_+ + \alpha^*\sigma_-)$$

$$U = e^{-\frac{i}{\hbar}H_I t} = \sum_{m=0}^{\infty}\frac{(-igt)^m}{m!}(\alpha\sigma_+ + \alpha^*\sigma_-)^m =$$

$$= \sum_{m=0}^{\infty}\frac{(-igt)^{2m}}{(2m)!}|\alpha|^{2m}1\!\!1 + \frac{(-igt)^{2m+1}}{(2m+1)!}(\alpha\sigma_+ + \alpha^*\sigma_-)|\alpha|^{\textcircled{2m}}$$

$$= \sum_{m=0}^{\infty}\underbrace{\overset{(-i)^{2m}}{\overset{\shortparallel}{(-1)^m}}\frac{(gt|\alpha|)^{2m}}{(2m)!}}_{\cos(gt|\alpha|)}1\!\!1 + \underbrace{\overset{(-i)^{2m}}{\overset{\shortparallel}{(-1)^m}}\frac{(-i)}{|\alpha|}\frac{(gt|\alpha|)^{2m+1}}{(2m+1)!}}_{=\sin(gt|\alpha|)}(\alpha\sigma_+ + \alpha^*\sigma_-)$$

$$= U = \cos(gt|\alpha|)\,1\!\!1 - i\sin(gt|\alpha|)\frac{\alpha\sigma_+ + \alpha^*\sigma_-}{|\alpha|}$$

$$U|\psi^I(0)\rangle = U|e_1\rangle = \cos(gt|\alpha|)|e_1\rangle - i\sin(gt|\alpha|)\times$$

$$\times\frac{\cancel{\alpha\sigma_+|e_1\rangle} + \alpha^*\overbrace{\sigma_-|e_1\rangle}^{|e_0\rangle}}{|\alpha|} = \cos(gt|\alpha|)|e_1\rangle \boxed{- i\sin(gt|\alpha|)\frac{\alpha^*}{|\alpha|}}|e_0\rangle$$

⇒ l'atomo OSCILLA tra st fondam e st eccitato

la prob di trovare l'atomo ~~eccitato~~ fondamentale è

$$P(e_0) = |\langle\psi|e_0\rangle|^2 = \underbrace{\left|-i\frac{\alpha^*}{|\alpha|}\right|^2}_{1} \boxed{\sin^{2} g t|\alpha|}$$

OSCILLAZIONI di RABI (RABI FLOPPING)

freq RABI $\overset{2|g\alpha|}{2\, g|\alpha|} = \Omega_R$

NON c'è oscillaz se $|\alpha| = 0$ → è SBAGLIATO

errore dovuto al fatto che abbiamo considerato il campo classico ⇒ le formule andranno bene per campi intensi $|\alpha| >> 1$