FISICA 2 16/12/20

Base per le forme differenziali $\to$ ${}^1 dx_{i,\vec{x}}$ dove x_i sono le componenti del $\vec{x}$ su una base di M

dim che $\frac{\partial}{\partial x_i}$ è una base per lo sp tg nel punto $\vec{x}$

deriv direz: $${}^1 df_{\vec{x}}(\vec{\xi}) \overset{def}{=} \left.\frac{d f(\vec{x}(t))}{dt}\right|_{\substack{t=0\\ \vec{x}}} = \sum_i \left.\frac{\partial f}{\partial x_i}\right|_{\vec{x}} \left.\frac{\partial x_i}{\partial t}\right|_{t=0}$$

$\vec{x} = \vec{x}(x_1(t), x_2(t))$

$$= \sum_i \left.\frac{\partial f}{\partial x_i}\right|_{\vec{x}} \xi_i = \left(\sum_i \xi_i \left.\frac{\partial}{\partial x_i}\right|_{\vec{x}} \right) f$$

$\vec{\xi}$ vett nello sp tg: $\vec{\xi} = \left.\frac{d\vec{x}(t)}{dt}\right|_{t=0} \iff \xi_i = \left.\frac{d x_i(t)}{dt}\right|_{t=0}$

cioè $\forall \vec{\xi}$ vett nello sp tg in $\vec{x}$ si può scrivere

$$\vec{\xi} = \sum_i \xi_i \left.\frac{\partial}{\partial x_i}\right|_{\vec{x}}$$

cioè $\left.\frac{\partial}{\partial x_i}\right|_{\vec{x}}$ è una base di $T_{\vec{x}} M$

la base forme differ $\{{}^1\omega_j\}$ t.c. ${}^1\omega_j(\vec{e}_i) = \delta_{ij}$

base

in questo caso ${}^1\omega_{j,\vec{x}}\left(\frac{\partial}{\partial x_i}\right) \overset{imp}{=} \delta_{ij}$

coord di $\vec{x}$ nella base

sappiamo $\frac{\partial(x_i)}{\partial x_j} = \delta_{ij}$

$\Rightarrow {}^1\omega_j$

$${}^1 df_{\vec{x}} = {}^1 dx_{j,\vec{x}}\left(\frac{\partial}{\partial x_i}\right) = \left.\frac{\partial x_j}{\partial x_i}\right|_{\vec{x}} = \delta_{ij}$$

$f = x_j$

base delle forme diff $\vec{x}$

$\forall$ forma diff $\quad {}^1\omega_{\vec{x}}(\vec{\xi}) = \sum_j \underbrace{\omega_j}_{\text{coeff.ic}} \; {}^1dx_{j,\vec{x}}(\vec{\xi})$

esempio: il differenziale

$${}^1df_{\vec{x}} = \sum_i \frac{\partial f}{\partial x_i} {}^1dx_i \overset{\text{coord cartes}}{=} \frac{\partial f}{\partial x}{}^1dx + \frac{\partial f}{\partial y}{}^1dy + \frac{\partial f}{\partial z}{}^1dz$$

f è scalare $\quad f = f(\vec{x}) = f(x_1, x_2 \dots)$

non tutte le forme differenziali sono dei differenziale

${}^1\omega = \omega_x \, {}^1dx + \omega_y \, {}^1dy$ non è un differenziale

se non esiste una fn t.c. $\frac{\partial f}{\partial x} = \omega_x$, $\frac{\partial f}{\partial y} = \omega_y$

cioè se non è vero che $\frac{\partial \omega_x}{\partial y} = \frac{\partial \omega_y}{\partial x}$

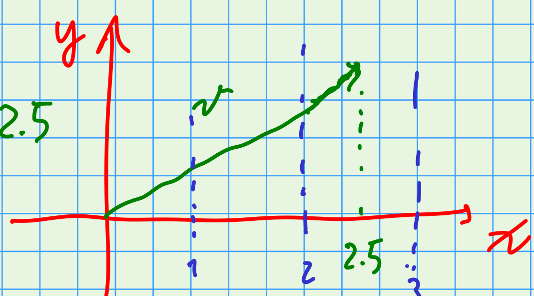
ESEMPI

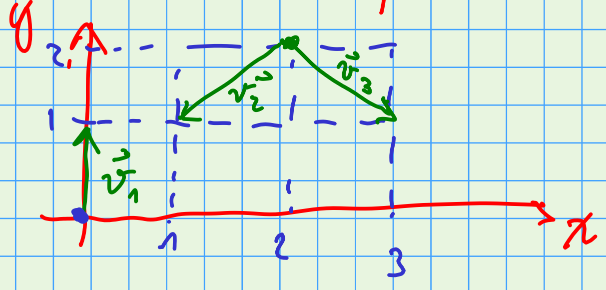
① coord cartes $\quad {}^1dx_1 = {}^1dx$, ${}^1dx_2 = {}^1dy$, ${}^1dx_3 = {}^1dz$

${}^1dx$: set di piani $\perp$ asse x

${}^1dy$: set di piani $\perp$ asse y

${}^1dx(\vec{v}) = 2.5$





${}^1\omega = {}^1dx$, $\alpha = x\,{}^1dy$, $\beta = dr^2$

$r^2 = x^2 + y^2$

ω: ${}^1dx(\vec{v}_1) = 0$

${}^1dx(\vec{v}_2) = -1 \qquad {}^1dx(\vec{v}_3) = 1$

α: $\underbrace{x}_{x}\,\underbrace{{}^1dy(\vec v_1)}_{{}^1dy(\vec v_1)} = 0 \cdot 1 = 0$ $\qquad$ $\underbrace{x}_{=2}\,\underbrace{{}^1dy}_{=-1}(\vec v_2) = -2$

$\underbrace{x}_{=2}\,\underbrace{{}^1dy}_{=-1}(\vec v_3) = -2$

$${}^1dr^2(\vec v_3) = 2\underbrace{x}_{=2}\,\underbrace{dx}_{=1} + 2\underbrace{y}_{=2}\,\underbrace{dy}_{=-1} = 0$$

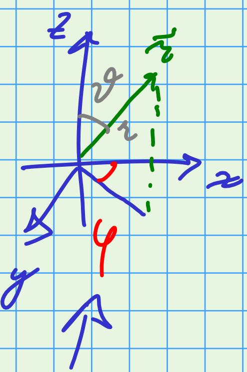
β: ${}^1dr^2 = {}^1d(x^2+y^2) = 2x\,{}^1dx + 2y\,{}^1dy$

${}^1dr^2(\vec v_1) = 0 \qquad {}^1dr^2(\vec v_2) = 2\underbrace{x}_{=2}\,\underbrace{{}^1dx}_{=-1} + 2\underbrace{y}_{=2}\,\underbrace{{}^1dy}_{=-1} = -8$

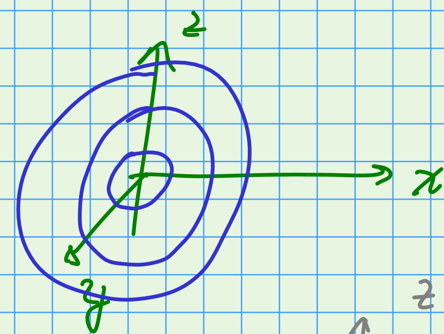
② coord polari sferiche

r, ϑ, φ

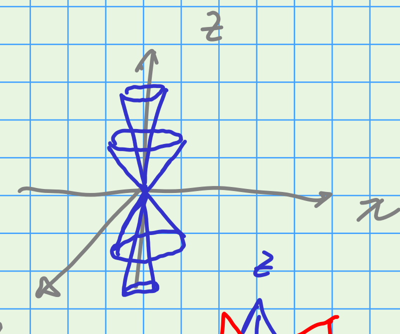
$$\begin{cases} x = r\sin\vartheta\cos\varphi \\ y = r\sin\vartheta\sin\varphi \\ z = r\cos\vartheta \end{cases}$$



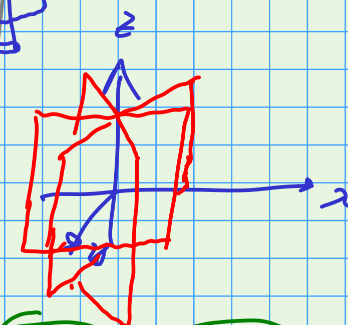
${}^1dr \to$ sfere concentriche



${}^1d\vartheta \to$ coni con vertice all'origine



${}^1d\varphi \to$ stella di piani centrata su asse z



${}^1dr, {}^1d\vartheta, {}^1d\varphi$ sono la base duale di $\dfrac{\partial}{\partial r}, \dfrac{\partial}{\partial \vartheta}, \dfrac{\partial}{\partial \varphi}$

NON sono la base duale dei versori $\vec e_r, \vec e_\vartheta, \vec e_\varphi$

la base duale di $\vec e_r, \vec e_\vartheta, \vec e_\varphi$

avevamo visto $d\vec r$ ha componente dr lungo $\vec e_r$

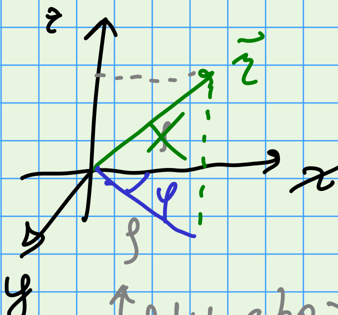
ha componente $r\,d\vartheta$ lungo $\vec{e}_\vartheta$

ha " $r\sin\vartheta\,d\varphi$ " $\vec{e}_\varphi$

$\Rightarrow$ ${}^1dr$, $r\,{}^1d\vartheta$, $r\sin\vartheta\,{}^1d\varphi$ base duale di $\vec{e}_r, \vec{e}_\vartheta, \vec{e}_\varphi$

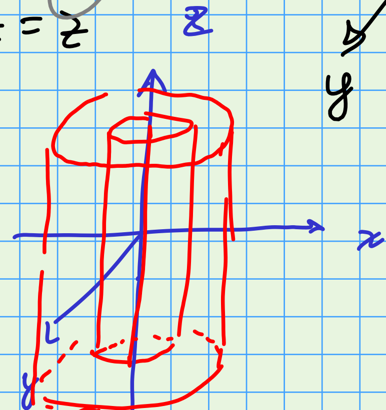
③ coord. cilindriche

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$



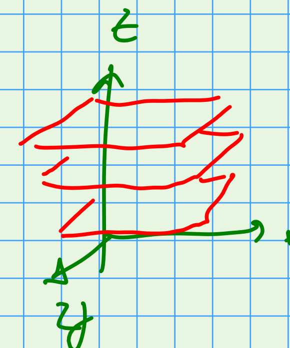
↑ ρ lunghezza del vettore proiettato sul piano x,y

${}^1d\rho$ → cilindri concentrici



${}^1d\varphi$ → stella piani con asse z

${}^1dz$ → piani paralleli ⊥ asse z



${}^1d\rho, {}^1d\varphi, {}^1dz$ sono base duale $\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z}$

${}^1d\rho, \rho\,{}^1d\varphi, {}^1dz$ " " " $\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z$

def derivata esterna → 1-forma nabla ${}^1\nabla$

$${}^1\nabla \overset{def}{=} \sum_i {}^1dx_i \frac{\partial}{\partial x_i}$$

è una 1-forma con. componenti le derivate risp x_i

$^1\nabla$ è indipendente dalle coordinate

$$^1\nabla \overset{def}{=} \boxed{\sum_i {}^1dx_i \frac{\partial}{\partial x_i}} = \sum_{ij} {}^1dx_i \overbrace{\frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial X_j}}^{\frac{\partial}{\partial x_i}} = \boxed{\sum_j {}^1dX_j \frac{\partial}{\partial X_j}}$$

cambio di coordinate

$$X_j = X_j(x_1 \dots x_d)$$

$$^1dX_j$$

$$^1dX_j\left(\frac{\partial}{\partial x_i}\right) = {}^1dX_j\left(\sum_k \frac{\partial X_k}{\partial x_i}\frac{\partial}{\partial X_k}\right) =$$

linearità

$$\sum_k \frac{\partial X_k}{\partial x_i} \underbrace{{}^1dX_j\left(\frac{\partial}{\partial X_k}\right)}_{\delta_{jk}} = \frac{\partial X_j}{\partial x_i}$$

$$^1dX_j = \sum_k {}^1dx_k \frac{\partial X_j}{\partial x_k}$$

ambo im danno lo stesso risultato quando agiscono sulla base $\frac{\partial}{\partial x_i}$

$\Rightarrow$ $^1\nabla$ è scritto nello stesso modo in tutte le basi, al contrario di $\vec{\nabla}$, ∇^μ

• derivata esterna di forme differ $^p\omega$

$$^1\nabla \wedge {}^p\omega = {}^{p+1}\gamma$$

forma di grado $p+1$ perché wedge product di 1-forma e p-forma $\Rightarrow$ $(p+1)$-forma

NOTAZIONE $\quad ^1\nabla\wedge = d$

$$d\,{}^p\omega = {}^1\nabla\wedge{}^p\omega$$

differenziale è una deriv esterna di una 0-Forma

$$\mathrm{d}\,{}^{0}\!f \overset{def}{=} {}^{1}\nabla \wedge {}^{0}\!f \overset{def}{=} \left(\sum_i {}^{1}\mathrm{d}x_i \frac{\partial}{\partial x_i}\right){}^{0}\!f \underset{\text{in coord cartes}}{=} {}^{1}\mathrm{d}x \frac{\partial f}{\partial x} + {}^{1}\mathrm{d}y \frac{\partial f}{\partial y} + {}^{1}\mathrm{d}z \frac{\partial f}{\partial z}$$

$$= \left(\sum_i {}^{1}\mathrm{d}X_i \frac{\partial}{\partial X_i}\right){}^{0}\!f$$

deriv esterna di 1-Forma:

$$\mathrm{d}\,{}^{1}\omega = {}^{1}\nabla \wedge {}^{1}\omega \underset{{}^{1}\omega = \sum_i \omega_i {}^{1}\mathrm{d}x_i}{=} {}^{1}\nabla \wedge \sum_i \omega_i \, {}^{1}\mathrm{d}x_i = \sum_{ij} \frac{\partial \omega_i}{\partial x_j} {}^{1}\mathrm{d}x_j \wedge {}^{1}\mathrm{d}x_i$$

identità differenziale $\boxed{\mathrm{d}^2 = 0} \Leftrightarrow$ rot grad = 0
div rot = 0

$$\mathrm{d}^2\,{}^{1}\omega = {}^{1}\nabla \wedge {}^{1}\nabla \wedge {}^{1}\omega \underset{\text{antisimmetria}}{=} 0 \quad \forall \omega$$

$${}^{1}a \wedge {}^{1}b\,(\vec{v}_1, \vec{v}_2) \overset{def}{=} \det \begin{pmatrix} {}^{1}a(\vec{v}_1) & {}^{1}b(\vec{v}_1) \\ {}^{1}a(\vec{v}_2) & {}^{1}b(\vec{v}_2) \end{pmatrix}$$

det è zero se due righe di una mat sono uguali

$$\underbrace{{}^{1}\nabla \wedge {}^{1}\nabla}_{=0} \wedge \ldots$$

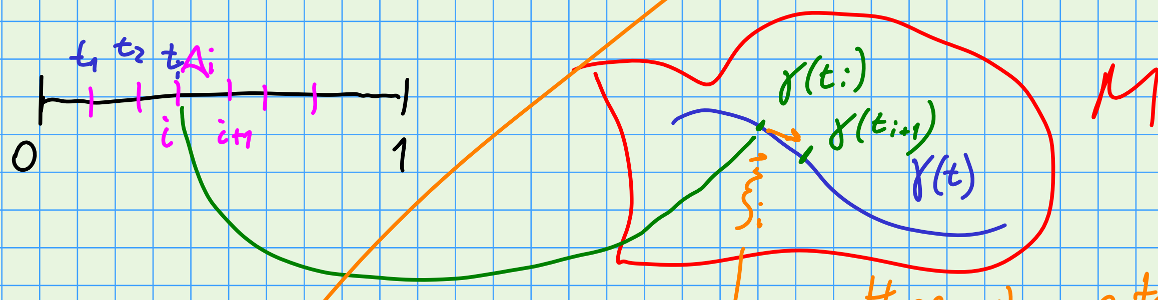
INTEGRAZIONE FORME DIFFERENZIALI

integraz di 1-forma ${}^{1}\omega$ lungo un cammino γ

sulla varietà M $\rightarrow$ parametrizzo $\gamma = \gamma(t)$ con $t \in [0,1]$

$$\int_\gamma {}^1\omega_{\vec{x}(t)} \stackrel{def}{=} \lim_{\max \Delta_i \to 0} \sum_i {}^1\omega_{\vec{x}}\left(\vec{\xi}_i\right)$$

divido l'intervallo $[0,1]$ in intervallini Δ_i



$\hookrightarrow$ vettore tangente a γ nel punto $\gamma(t_i)$ lungo $\gamma(t_i) - \gamma(t_{i+1})$

nel $\lim_{\max \Delta_i \to 0}$ $\quad \vec{\xi}_i \rightarrow$ approssima il tratto $\gamma(t_i)$

diventano vettori infinitesimi

$${}^1\omega_{\vec{x}}\left(\vec{\xi}_i\right) = {}^1\omega_{\gamma(t_i)}\left(\vec{\xi}_i\right) = \sum_k \omega_k \underbrace{{}^1dx_k\left(\vec{\xi}_i\right)}_{\xi_k} =$$

$$= \sum_k \omega_k \left(\vec{\xi}_i\right)_k \sim \sum_k \omega_k \underbrace{\left(\gamma(t_{i+1}) - \gamma(t_i)\right)_k}_{\Delta_i} \Delta_i =$$

$$= \sum_k \omega_k \frac{\partial \gamma_k}{\partial t} dt \longrightarrow \quad \sum_k \rightarrow \int dt$$

$$\int_\gamma {}^1\omega = \int_0^1 dt \sum_k \omega_k \frac{\partial \gamma_k}{\partial t} \stackrel{\text{coord. cartes}}{=} \int_0^1 dt \left(\omega_x(\gamma(t)) \frac{\partial \gamma_x}{\partial x} + \right.$$

$$\omega_{k,\vec{x}} = \omega_k(\gamma(t))$$

$$+ \underbrace{\omega_y(\gamma(t))\frac{\partial\gamma_y}{\partial y} + \omega_z(\gamma(t))\frac{\partial\gamma_z}{\partial z}}\Big) = \int_0^1 dt\ \vec{\omega}\cdot\vec{n} = \int_\gamma d\vec{r}\cdot\vec{\omega}$$

$$\vec{n} \overset{def}{=} \left(\frac{\partial\gamma_x}{\partial t}, \frac{\partial\gamma_y}{\partial t}, \frac{\partial\gamma_z}{\partial t}\right)$$

$$= \int_\gamma {}^1\omega$$

è l'integrale di circuitaz del vett duale $\vec{\omega}$

integraz di n-forma definita su $\mathbb{R}^k$

$$\int_{\substack{V \\ \text{dim } k}} {}^k\omega \overset{def}{=} \int_V dx_1\, dx_2 \dots dx_n\ \varphi(\vec{x})$$

$\omega(\vec{x})$ è il coefficiente ${}^k\omega = (dx_1 \wedge dx_2 \dots dx_n)\varphi$

${}^k\omega$ su $\mathbb{R}^k$ ha dim 1

$$\dim = \binom{k}{n} = 1$$

integrale di k-forme su varietà a dim $d \geq k$ integrato su una sottovarietà di dim k

$$\int_{\substack{S \\ \text{dim } k}} {}^k\omega =$$

parametrizzo S in termini di k variabili e poi mi riconduco al caso precedente caso 2-forme in $\mathbb{R}^3$ integrate su S dim 2

$$\int_S \overset{2}{\omega} = \int_S \overset{1}{dx} \wedge \overset{1}{dy}\, A + \overset{1}{dy} \wedge \overset{1}{dz}\, B + \overset{1}{dz} \wedge \overset{1}{dx}\, C$$

in coord cartes

dove A, B, C componenti di $\overset{2}{\omega}$ su $\overset{1}{dx_i} \wedge \overset{1}{dx_j}$

$$\overset{2}{\omega} = A\, \underbrace{\overset{1}{dx} \wedge \overset{1}{dy}} + B\, \underbrace{\overset{1}{dy} \wedge \overset{1}{dz}} + C\, \underbrace{\overset{1}{dz} \wedge \overset{1}{dx}}$$

= integrale in termini di u, v → domani

parametrizzo S in termini di 2 variabili $S = S(u,v)$